A Unified Approach to Portfolio Optimization with Linear Transaction Costs

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Abstract

In this paper we study the continuous time optimal portfolio selection problem for an investor with a finite horizon who maximizes expected utility of terminal wealth and faces transaction costs in the capital market. It is well known that, depending on a particular structure of transaction costs, such a problem is formulated and solved within either stochastic singular control or stochastic impulse control framework. In this paper we propose a unified framework, which generalizes the contemporary approaches and is capable to deal with any problem where transaction costs are a linear/piecewise-linear function of the volume of trade. We also discuss some methods for solving numerically the problem within our unified framework.

Key words: portfolio choice, transaction costs, stochastic singular control, stochastic impulse control, computational methods.

JEL classification: C61, C63, G11.

1 Introduction

In this paper we study the continuous time optimal portfolio selection problem for an investor with a finite horizon who maximizes expected utility of terminal wealth and faces transaction costs in the capital market. It is well known that, depending on a particular structure of transaction costs, such a problem is formulated and solved within either classical stochastic control, stochastic singular control, or stochastic impulse control framework. The purpose of this paper is to suggest a unified theoretical framework, which generalizes the contemporary approaches and is capable to deal with any problem where transaction costs are a linear/piecewise-linear function of the amount of the risky asset traded.

The asset allocation problem we consider is a variant of the classical consumption-investment problem in modern finance. In the absence of transaction costs, applying the theory of (classical) optimal stochastic control, the closed-form solutions for some particular utility functions were obtained by Merton (see, for example, Merton (1971)).

The introduction of transaction costs adds considerable complexity to the optimal portfolio selection problem. The problem is simplified if one assumes that the transaction costs are proportional to the amount of the risky asset traded, and there are no transaction costs on trades in the riskless asset. In this case the problem was solved by Davis and Norman (1990) applying the theory of stochastic singular control. Their work was further extended by Shreve and Soner (1994), Akian, Menaldi, and Sulem (1996), Framstad, Øksendal, and Sulem (2001), and many others.

The solution of the optimal portfolio selection problem with a fixed cost component is based on the theory of stochastic impulse control. The first application of this theory to a consumption-investment problem was done by Eastham and Hastings (1988). This initial work was extended by Hastings (1992), Korn (1998), Øksendal and Sulem (2002), and some others.

The main goal of this paper is to try to integrate the stochastic singular and impulse control approaches into a single approach, which will allow to formulate and solve the above mentioned problems within a unified framework. In short, our idea is to consider the optimal portfolio choice problem with a linear structure of transaction costs as a stochastic singular control problem and regard the stochastic impulse control as a special case of a singular stochastic control where all controls are discontinuous. On the contrary, the general formulation of the problem and the solution technique were inspired mainly by the stochastic impulse control theory.

Below in this section we present a general formulation of the optimal portfolio choice problem with transaction costs. The rest of the paper is organized as follows. Section 2 presents the standard formulations of the problem in the presence of proportional transaction costs only, and in the presence of both fixed and proportional transaction costs. In Section 3 we propose a unified framework for the optimal portfolio choice problem with a linear structure of transaction costs. Section 4 outlines some methods for solving numerically the optimal portfolio selection problem within this unified framework. Finally, Section 5 concludes the paper and discusses some possible extensions.

Throughout the paper we consider a continuous-time economy with one risky and one risk-free asset. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a given filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$. The risk-free asset, which we will refer to as the bank account, pays a constant interest rate of $r \geq 0$, and, consequently, the evolution of the amount invested in the bank, x_t, is given by the ordinary differential equation

$$dx_t = rx_t dt. \tag{1.1}$$

We will refer to the risky asset to as the stock, and assume that the price of the stock, S_t, evolves according to a geometric Brownian motion defined by

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1.2}$$

where μ and σ are constants, and B_t is a one-dimensional $\mathcal{F}_t$-Brownian motion.

The investor holds x_t in the bank account and the amount y_t in the stock at time t. We suppose that at any time the investor can decide to transfer money from the bank account to the stock and conversely. We assume that a purchase or sale of stocks of the amount ξ incurs transaction costs $f(\xi)$, the structure of which we will specify later. These costs are drawn from the bank account.

If the investor has the amount x_t in the bank account, and the amount y_t in the stock, his *net wealth*, which we denote as X_t, is defined as the holdings in the bank account after either selling of all shares of the stock (if the proceeds are positive after transaction costs) or closing of the short position in the stock and is given by

$$X_t(x,y) = \begin{cases} \max\{x_t + y_t - f(y_t), x_t\} & \text{if } y_t \geq 0, \\ x_t + y_t - f(y_t) & \text{if } y_t < 0. \end{cases} \tag{1.3}$$

We consider an investor with a finite horizon $[0,T]$ who has utility only of terminal wealth. The simplest investor's problem is to choose an admissible trading strategy to maximize $E_t[U(X_T)]$, i.e., the expected utility of his net terminal wealth, subject to the self-financing constraint. We define the value function at time t as

$$V(t,x,y) = \sup_{\mathcal{A}(x,y)} E_t^{x,y}[U(X_T)], \tag{1.4}$$

where $\mathcal{A}(x,y)$ denotes the set of admissible controls available to the investor who starts at time t with an amount of x in the bank and y holdings in the stock. This basic problem can be extended by introducing: European-style derivatives as in Hodges and Neuberger (1989), and American-style derivatives as in Davis and Zariphopoulou (1995).

Generally, one requires that the state process (x,y) should remain in some region $\mathcal{S} \subset \mathbb{R}^2$. That is

$$(x_t, y_t) \in \mathcal{S} \quad \forall \quad t.$$

Therefore, the set of admissible controls, which transfer the state (x,y) into the state (x', y'), is defined by

$$\mathcal{A}(x,y) = \{(x,y) \in \mathcal{S} : (x', y') \in \mathcal{S}\}.$$

For example, for some utility functions it is necessary to define a *solvency region* which is the set of states where the net wealth is nonnegative:

$$\mathcal{S} = \{(x,y) \in \mathbb{R}^2 : X(x,y) \geq 0\}. \tag{1.5}$$

The solution to the problem (1.4) gives the value function and, above all, the optimal portfolio policy the investor should follow in order to attain the maximum expected utility.

2 The Standard Formulations of the Problem with Different Structures of Transaction Costs

In this section we present the standard formulations of the problem in the presence of only proportional transaction costs, and in the presence of both fixed and proportional transaction costs. Note that the problem in the market where each transaction has only a fixed costs component is also formulated in the same manner as that with both fixed and proportional transaction costs, that is, within the framework of the impulse control theory. The formulation of the problem in the absence of transaction costs could be obtained as the limiting case of the problem with only proportional transaction costs when the rate of proportional transaction costs goes to zero.

2.1 Proportional Transaction Costs

Our presentation here is due to Davis and Norman (1990), who were the first to solve this problem in continuous time. Since then this presentation became a standard when it comes to the optimal portfolio choice problem with proportional transaction costs only.

We assume that a purchase or sale of stocks of the amount ξ incurs transaction costs $f(\xi) = \lambda|\xi|$ proportional to the transaction ($\lambda > 0$). The evolution equations for the system (x_t, y_t) are

$$\begin{aligned} dx_t &= rx_t dt - (1+\lambda)dL_t + (1-\lambda)dM_t, \\ dy_t &= \mu y_t dt + \sigma y_t dB_t + dL_t - dM_t, \end{aligned} \tag{2.1}$$

where L_t, M_t represent cumulative purchase and sale, respectively, of the stock up to time t. Both L_t and M_t are right-continuous with left-hand limits (RCLL) nonnegative and nondecreasing $\{\mathcal{F}_t\}$-adapted processes. By convention, $L_0 = M_0 = 0$.

In contrast to the no transaction cost case, where the optimal policy requires continuous rebalancing, at any time t the portfolio space is divided into three disjoint regions, which can be specified as the Buy region, the Sell region, and the No-Transaction (NT) region, and two boundaries describe the optimal policy. If a portfolio lies either in the Buy region or in the Sell region, the optimal strategy is to buy/sell the risky asset until the portfolio reaches the closest boundary of the NT region. If a portfolio lies in the NT region, it is not adjusted at that time.

In the literature, there are two main approaches to formulation of the problem in terms of HJB inequalities. Using the first approach, one derives the associated HJB inequalities[1] by giving heuristic arguments. This approach gives clear insights into the nature of the problem, but suffers from the lack of rigor. Using the second approach, one presents some HJB inequalities and then, by means of a so-called verification theorem, proves that a function, which satisfies these HJB inequalities, is indeed the value function of the problem. Even though it is a rigorous treatment of the problem, the reader often lacks understanding of useful insights into the problem. Alternatively, as we present below, one combines these two approaches.

For example, the heuristic arguments, which lead to the HJB inequalities, could be as follows: If for some initial point (t, x, y) the optimal strategy is to not transact, the utility associated with this strategy is $V(t, x, y)$. The necessary conditions for the optimality of the no transaction strategy is $-(1+\lambda)V_x + V_y \leq 0$ and $(1-\lambda)V_x - V_y \leq 0$. That is, it is not possible for the investor to increase his indirect utility function by either buying or selling some infinitesimal amount of the stock at the expense of lowering or increasing, respectively, the holdings in the bank account. These inequalities hold with equalities when it is optimal to rebalance the portfolio. The set of (x, y) points for which $-(1+\lambda)V_x+V_y = 0$ defines the Buy region. Similarly, the equation $(1-\lambda)V_x - V_y = 0$ defines the Sell region. Moreover, in the NT region, the application of the dynamic programming principle[2] gives $\mathcal{L}V(t, x, y) = 0$, where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}V(t, x, y) = V_t + rxV_x + \mu yV_y + \frac{1}{2}\sigma^2 y^2 V_{yy}. \tag{2.2}$$

As a consequence, the value function V defined by (1.4) is a solution of the Hamilton-Jacobi-Bellman inequalities:

$$\max\Big\{\mathcal{L}V, \quad -(1+\lambda)V_x + V_y, \quad (1-\lambda)V_x - V_y\Big\} = 0 \tag{2.3}$$

[1] They represent an infinitesimal version of the principle of dynamic programming applied to the problem under investigation.

[2] Note that here the process (x, y) becomes an uncontrolled diffusion.

with the boundary condition

$$V(T,x,y) = U(X_T).$$

Then, by means of a verification theorem, we can prove that a function which satisfies HJB inequalities (2.3) is indeed the value function defined by (1.4). A sketch of such a theorem, with a proof, is given below.

Theorem 2.1 (Verification Theorem 1). *Suppose there exists a function $v(t,x,y) \in C^{1,1,2}$ that satisfies the growth conditions and an admissible control $(L(t), M(t))$ such that*

$$\max\Big\{\mathcal{L}v,\ -(1+\lambda)v_x + v_y,\ (1-\lambda)v_x - v_y\Big\} = 0 \quad \textit{on } [0,T]\times\mathcal{S},$$

$$\begin{aligned} [-(1+\lambda)v_x + v_y]dL^c(t) &= 0 \quad \textit{for all } t, \\ [(1-\lambda)v_x - v_y]dM^c(t) &= 0 \quad \textit{for all } t, \end{aligned} \tag{2.4}$$

$$\mathcal{L}v(t,x,y) = 0 \quad \textit{when } dL(t) = 0 \textit{ and } dM(t) = 0, \tag{2.5}$$

$$v(T,x,y) = U(X_T), \tag{2.6}$$

then

$$v(t,x,y) = V(t,x,y),$$

and the control $(L(t), M(t))$ is optimal.

Remark 2.1. $L^c(t)$ and $M^c(t)$ denote the continuous parts of $L(t)$ and $M(t)$, respectively:

$$\begin{aligned} L^c(t) &\stackrel{\text{def}}{=} L(t) - \textstyle\sum_{0\le s\le t}(L(s) - L(s-)), \\ M^c(t) &\stackrel{\text{def}}{=} M(t) - \textstyle\sum_{0\le s\le t}(M(s) - M(s-)). \end{aligned}$$

The no transaction region D is defined by

$$D = \{(x,y) \in \mathcal{S} : -(1+\lambda)v_x + v_y < 0 \text{ and } (1-\lambda)v_x - v_y < 0\}. \tag{2.7}$$

Proof. Using the Ito's rule for semimartingales we obtain

$$E_t^{x,y}[v(T,x,y)] = v(t,x_t,y_t) + \int_t^T \mathcal{L}v(s,x_s,y_s)ds + \int_t^T [-(1+\lambda)v_x + v_y]dL^c(s) \\ + \int_t^T [(1-\lambda)v_x - v_y]dM^c(s) + \sum_{t\le s\le T} [v(s,x_s,y_s) - v(s-,x_{s-},y_{s-})]. \tag{2.8}$$

The last term in (2.8) is due to the discontinuities of (x,y) caused by application of a (non-infinitesimal) control in the Buy or the Sell region[3]. We know that in the Buy or the Sell region the value function remains constant along the path of the state dictated by the optimal trading strategy:

$$\begin{aligned} v(t-,x,y) &= v(t,x-(1+\lambda)\Delta y, y+\Delta y) \quad \text{in the Buy region,} \\ v(t-,x,y) &= v(t,x+(1-\lambda)\Delta y, y-\Delta y) \quad \text{in the Sell region,} \end{aligned} \tag{2.9}$$

where Δy, the amount of the stock either bought or sold by the investor, can take any positive value up to the quantity required to take the state to the closest boundary of the NT region. Therefore, the last term in (2.8) equals to zero. Finally, using (2.4), (2.5), (2.9), and (2.6) we get

$$v(t,x,y) = E_t^{x,y}[v(T,x,y)] = V(t,x,y). \qquad \square$$

For a rigorous version of the same theorem, see, for example, Framstad et al. (2001).

Remark 2.2. If we apply a not optimal control, then, as it follows from HJB inequalities (2.3) and Theorem 2.1, we get

$$\begin{aligned} (-(1+\lambda)v_x + v_y)dL^c(t) &< 0, \\ ((1-\lambda)v_x - v_y)dM^c(t) &< 0, \\ \mathcal{L}v(t,x,y) &< 0, \\ v(s,x_s,y_s) - v(s-,x_{s-},y_{s-}) &< 0. \end{aligned}$$

All this gives

$$E_t^{x,y}[v(T,x,y)] = v(t,x,y) - \varepsilon, \tag{2.10}$$

[3]When the stock follows a diffusion process, the control is continuous except, possibly, the initial jump to the nearest boundary of the NT region if the portfolio lies outside of the NT region.

where ε is some positive value. As we want to maximize the left hand side of (2.10), it could only be attained when $\varepsilon = 0$, that is, when the control is optimal.

The main result of the verification theorem is that any function $v \in C^{1,1,2}$ which satisfies the given requirements is necessarily the value function of the corresponding optimal portfolio choice problem. The problem is that one cannot guarantee sufficient regularity of the solution to (2.3). Nevertheless, it turns out that the solution does not need to be $C^{1,1,2}$ if we interpret these equations in an appropriate weak sense. This weak solution concept was introduced by Crandall and Lions and is called *viscosity* solution (standard references are Crandall, Ishii, and Lions (1992) and Fleming and Soner (1993)). As a converse to the verification theorem it is now customary to prove the viscosity property of the value function.

Theorem 2.2. *The value function V defined by (1.4), assuming it is continuous in $[0,T] \times \mathcal{S}$, is a viscosity solution of (2.3).*

For a proof of the theorem see, for example, Davis, Panas, and Zariphopoulou (1993) Theorem 2.

2.2 Fixed and Proportional Transaction Costs

Our presentation here is due to Øksendal and Sulem (2002).

We assume that a purchase or sale of stocks of the amount ξ incurs a transaction costs consisting of a sum of a fixed cost $k > 0$ (independent of the size of transaction) plus a cost $\lambda|\xi|$ proportional to the transaction, so that $f(\xi) = k + \lambda|\xi|$, $\lambda \geq 0$. The control of the investor is a so-called *impulse control* $\eta = (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)$. Here $0 \leq \tau_1 < \tau_2 < \ldots$ are $\mathcal{F}_t$-stopping times giving the times when the investor decides to change his portfolio, and ξ_j are $\mathcal{F}_{\tau_j}$-measurable random variables giving the sizes of the transactions at these times. If such a control is applied to the system (x_t, y_t), it gets the form

$$\begin{aligned}
&dx_t = rx_t dt; && \tau_i \leq t < \tau_{i+1} \\
&dy_t = \mu y_t dt + \sigma y_t dB_t; && \tau_i \leq t < \tau_{i+1} \\
&x_{\tau_{i+1}} = x_{\tau_{i+1}^-} - k - \xi_{i+1} - \lambda|\xi_{i+1}|, && \\
&y_{\tau_{i+1}} = y_{\tau_{i+1}^-} + \xi_{i+1}. &&
\end{aligned} \tag{2.11}$$

One defines the *intervention operator* (or the maximum utility operator)

$\mathcal{M}$ by

$$\mathcal{M}V(t,x,y) = \sup_{(x',y')\in\mathcal{A}(x,y)} V(t,x',y'), \tag{2.12}$$

where x' and y' are the new values[4] of x and y. In other words, $\mathcal{M}V(t,x,y)$ represents the value of the strategy that consists in choosing the best transaction. One defines the no transaction region D by

$$D = \big\{(x,y) : V(t,x,y) > \mathcal{M}V(t,x,y)\big\}. \tag{2.13}$$

As in the case with proportional transaction costs only, in the presence of both fixed and proportional transaction costs at any time t the portfolio space is divided into three disjoint regions: Buy, Sell, and NT. However, the optimal policy in this case is described by four boundaries. The Buy and NT regions are divided by the lower no-transaction boundary, and the Sell and NT regions are divided by the upper no-transaction boundary. If a portfolio lies in the Buy region, the optimal strategy is to buy the risky asset until the portfolio reaches the Buy target boundary. Similarly, if a portfolio lies in the Sell region, the optimal strategy is to sell the risky asset until the portfolio reaches the Sell target boundary.

Again, the heuristic arguments intended to characterize the value function and the associated optimal strategy could be as follows: If for some initial point (t,x,y) the optimal strategy is to not transact, the utility associated with this strategy is $V(t,x,y)$. Choosing the best transaction and then following the optimal strategy gives the utility $\mathcal{M}V(t,x,y)$. The necessary condition for the optimality of the no transaction strategy is $V(t,x,y) \geq \mathcal{M}V(t,x,y)$. This inequality holds with equality when it is optimal to rebalance the portfolio. Moreover, in the NT region, the application of the dynamic programming principle gives $\mathcal{L}V(t,x,y) = 0$, where the operator $\mathcal{L}$ is defined by (2.2).

Consequently, the value function V defined by (1.4) is a solution of the so-called quasi-variational Hamilton-Jacobi-Bellman inequalities (QVHJBI, or just QVI):

$$\max\Big\{\mathcal{L}V, \quad \mathcal{M}V - V\Big\} = 0, \tag{2.14}$$

[4] That is, $y' = y + \Delta y$ and $x' = x - k - \Delta y - \lambda|\Delta y|$, where Δy is the size of transaction.

with the boundary condition

$$V(T,x,y) = U(X_T).$$

Now, by means of a verification theorem, we can prove that a function, which satisfies QVI (2.14), is indeed the value function defined by (1.4). Below we present a sketch of such a theorem, with a proof. A more rigorous formulation of the same theorem and a proof one can find in, for example, Øksendal and Sulem (2002).

Theorem 2.3 (Verification Theorem 2). *Suppose there exists a function $v(t,x,y) \in C^{1,1,2}$ that satisfies the growth conditions and an admissible control $\eta = (\tau_1, \tau_2, \ldots; \xi_1, \xi_2, \ldots)$ such that*

$$\max\Big\{\mathcal{L}v, \quad \mathcal{M}v - v\Big\} = 0 \quad \text{on } [0,T] \times \mathcal{S},$$

$$\mathcal{M}v - v = 0 \quad \text{outside } D, \tag{2.15}$$

$$\mathcal{L}v(t,x,y) = 0 \quad \text{in } D, \tag{2.16}$$

$$v(T,x,y) = U(X_T), \tag{2.17}$$

then

$$v(t,x,y) = V(t,x,y)$$

and the control η, given by

$$\begin{aligned} \tau_i &= \inf\{t > \tau_{i-1} : (t,x,y) \notin D\}, \\ \xi_i &:= \arg\max\big\{v(\tau_i, x - k - \xi_i - \lambda|\xi_i|, y + \xi_i)\big\}, \end{aligned}$$

is optimal.

Proof. Using the classical Ito's rule between the stopping times τ_i when the control ξ_i is applied[5], we obtain

$$\begin{aligned} E_t^{x,y}[v(T,x_T,y_T)] = v(t,x_t,y_t) &+ \int_t^T \mathcal{L}v(s,x_s,y_s)ds \\ &+ \sum_{t\le s\le T} [v(s,x_s,y_s) - v(s-,x_{s-},y_{s-})]. \end{aligned} \tag{2.18}$$

The last term in (2.18) represents the change in the value function when

[5] Note that the control is applied when the portfolio process goes out of the NT region.

some control is applied (assuming $s \equiv \tau_i$). First, note that the transactions are maid in order to maximize the expected utility, that is $v(s, x_s, y_s) = \mathcal{M}v(s-, x_{s-}, y_{s-})$. Then, using (2.15) we obtain

$$\sum_{t \leq s \leq T} [v(s, x_s, y_s) - v(s-, x_{s-}, y_{s-})] = \sum_{t \leq s \leq T} [\mathcal{M}v(s-, x, y) - v(s-, x, y)] = 0.$$

Between the transactions, the portfolio lies inside the NT region. This means, according to (2.16), that the second term in (2.18) is also equal to zero. That is,

$$\int_t^T \mathcal{L}v(s, x, y)ds = 0.$$

Finally, using (2.17) we get

$$v(t, x, y) = E_t^{x,y}[v(T, x, y)] = V(t, x, y). \qquad \square$$

Another way to attack the impulse control problem via an iterative sequence of optimal stopping problems was suggested by Korn (1998). By noting that when it is optimal to make a transaction $V(\tau, x, y)$ and $\mathcal{M}V(\tau, x, y)$ must coincide, we conjecture that the following dynamic programming principle is to hold

$$V(t, x, y) = \max \left\{ \sup_{\tau \in \Theta} E_t[\mathcal{M}V(\tau, x(\tau), y(\tau))], \quad V_0(t, x, y) \right\},$$

where Θ is the set of finite stopping times, and $V_0(t, x, y)$ is the expected utility of the no transaction strategy starting in (t, x, y). This method can be implemented to solve numerically the QVI.

As a converse to the verification theorem one can prove the viscosity property of the value function.

Theorem 2.4. *The value function V defined by (1.4), assuming it is continuous in $[0, T] \times \mathcal{S}$, is a viscosity solution of (2.14).*

For a proof of the theorem see, for example, Korn (1999) Theorem 4.2.

3 A Unified Approach to the Problem when Transaction Costs are Linear

In the first part of this section we give some reasons which motivate the search for a unified approach to the optimal portfolio selection problem with a linear structure of transaction costs. In the second part we suggest a unified framework for the optimal portfolio selection problem when transaction costs are linear/piecewise-linear in the size of trade.

3.1 Preliminary Discussion

By linear structure of transaction costs we mean the following specification of the transaction costs as a function of the volume of trade

$$f(d\xi) = I_{\{d\xi \neq 0\}} k + \lambda |d\xi|, \tag{3.1}$$

where $k \geq 0$ and $\lambda \geq 0$, and $I_{\{d\xi \neq 0\}}$ is an indicator function defined by

$$I_{\{d\xi \neq 0\}} = \begin{cases} 1 & \text{if } d\xi \neq 0, \\ 0 & \text{if } d\xi = 0. \end{cases}$$

First of all, it looks like that the optimal portfolio choice problems with and without a fixed cost component require their own formulation and particular solution method. Nevertheless, working mainly with numerical solutions of all these problems we discovered that a single general numerical algorithm, intended to work on the problem with both fixed and proportional transaction costs, is capable to solve a problem with any linear transaction costs structure. Indeed, the common sense tells us that as $k \to 0$ the solution of, for example, the optimal portfolio choice problem with both fixed and proportional transaction costs should converge to the solution of the optimal portfolio choice problem with proportional transaction costs only.

Second, when it comes to the numerical computation of the value function and the associated optimal policy of the problem with only proportional transaction costs, the two inequalities in (2.3), which describe the Buy and the Sell region: (i) do not reflect the maximization nature of the problem and (ii) cannot be implemented *explicitly* in a numerical method. The catch is that these two inequalities[6] describe how the value function should behave provided we know the value function at, say, times t and $t + dt$. On the contrary, any numerical method, either a finite-difference or a Markov chain approximation, implements a dynamic programming algorithm where the unknown values at time t is found by using the known values at the next time instant $t + dt$. Thus, these inequalities provide only an implicit indication on how to compute the value function.

Let us elaborate on this more specifically. Consider the value function V of the problem with proportional transaction costs only. Assume we know the value function V at time $t + dt$. How do we proceed to find the value function at time t? An obvious start is to solve the partial differential equation $\mathcal{L}V(t, x, y) = 0$ between times t and $t + dt$ to find a lower[7] estimate for the value function. Then one finds the NT region where both $-(1 + \lambda)V_x + V_y < 0$ and $(1 - \lambda)V_x - V_y < 0$ are satisfied. Outside the NT region the value function is recomputed by using

$$V(t, x, y) = \begin{cases} V(t, x + (1 - \lambda)(y - y_u), y_u) & \text{if } (x, y) \in \text{Sell region}, \\ V(t, x - (1 + \lambda)(y_l - y), y_l) & \text{if } (x, y) \in \text{Buy region}, \end{cases}$$

where y_u and y_l are points on the upper and lower boundaries, respectively, of the NT region. This follows from the optimal transaction policy which mandates to transact to the nearest boundary of the NT region if the portfolio lies outside this region.

A possible problem with such an algorithm is that there might be several regions where both $-(1 + \lambda)V_x + V_y < 0$ and $(1 - \lambda)V_x - V_y < 0$ are satisfied. This could happen in the case when the lower estimate of the value function, after solving the partial differential equation $\mathcal{L}V(t, x, y) = 0$, has multiple local maxima[8] which, in their turn, produce multiple maxima as one transacts along the Buy or Sell direction. In this situation we face the problem of choosing correct NT sub-regions and transaction policy. We encountered this problem when we calculated the value function of the buyer

[6] These inequalities may alternatively be called as gradient constraints.

[7] That is, we find the expectation of the value function at the next time instant. Generally, the value function must be not less than this expectation: $V(t, x(t), y(t)) \geq E\{V(t + dt, x(t + dt), y(t + dt))\}$

[8] Note that the conditions $-(1 + \lambda)V_x + V_y = 0$ and $(1 - \lambda)V_x - V_y = 0$ are nothing else than the first order conditions of a local extremum as one transacts along the Buy or Sell direction, respectively.

of an American option using the model of Davis and Zariphopoulou (1995), see Zakamouline (2003). In this situation the only way to overcome such a problem is to perform an explicit search of a global maximum.

The inequalities for the Buy and the Sell regions tell us that it is impossible to increase the value function by either buying or selling some amount of the stock at the expense of lowering or increasing, respectively, the holdings in the bank account. An alternative and more explicit numerical procedure to solve the optimal portfolio selection problem with proportional transaction cost is analogous to that used to solve the optimal portfolio selection problem with both fixed and proportional transaction costs. As before, we start with solving the partial differential equation $\mathcal{L}V(t,x,y) = 0$ for the no-transaction problem. Then we need to compare the value function at each point (x, y) with the maximum attainable values from either buying or selling some amount of the stock. Mathematically this procedure is described by the maximum utility operator $\mathcal{M}$.

Third. The same heuristic arguments, intended to characterize the value function and the associated optimal strategy for the optimal portfolio choice problem with both fixed and proportional transaction costs, may be *formally* applied for the case with proportional transaction costs only. That is, an equivalent characterization of the value function for the problem in the presence of only proportional transaction costs could be expressed by means of QVI (2.14). Moreover, we can prove that the two different formulations of the same problem, (2.3) and (2.14), yield the same result as concerns the optimal portfolio selection problem with proportional transaction costs only. It suffices to prove the following theorem:

Theorem 3.1. *For the optimal portfolio selection problem with only proportional transaction costs,*

$$\begin{aligned} -(1+\lambda)V_x + V_y \leq 0, \\ (1-\lambda)V_x - V_y \leq 0, \end{aligned} \tag{3.2}$$

if and only if

$$\mathcal{M}V - V \leq 0. \tag{3.3}$$

Proof. The first part. Assume (3.2) holds. Chose any point (x_0, y_0). Suppose that the maximum along the Buy line starting in (x_0, y_0) is attained at the point $(x_0 - (1+\lambda)\alpha, y_0 + \alpha)$, and that the maximum along the Sell line starting in (x_0, y_0) is attained at the point $(x_0 + (1-\lambda)\beta, y_0 - \beta)$. Then for the maximum along the Buy line we have that

$$
\begin{aligned}
& V(x_0 - (1+\lambda)\alpha, y_0 + \alpha) = V(x_0, y_0) \\
& + \int_0^\alpha \big[- (1+\lambda) V_x(x_0 - (1+\lambda)s, y_0 + s) + V_y(x_0 - (1+\lambda)s, y_0 + s) \big] ds \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le V(x_0, y_0).
\end{aligned}
$$

Similarly, for the maximum along the Sell line we have that

$$
\begin{aligned}
& V(x_0 + (1-\lambda)\beta, y_0 - \beta) = V(x_0, y_0) \\
& \quad + \int_0^\beta \big[(1-\lambda) V_x(x_0 + (1-\lambda)s, y_0 + s) - V_y(x_0 + (1-\lambda)s, y_0 + s) \big] ds \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \le V(x_0, y_0).
\end{aligned}
$$

Consequently, $\mathcal{M}V(x_0, y_0) - V(x_0, y_0) \le 0$. Since the point (x_0, y_0) was chosen arbitrary, this holds for every point (x, y) in the domain of V.

The second part. Assume (3.3) holds. Chose any point (x_0, y_0). Then for any point[9] along the Buy line starting in (x_0, y_0) we have that

$$
V(x_0 - (1+\lambda)h, y_0 + h) \le V(x_0, y_0),
$$

and for any point along the Sell line starting in (x_0, y_0) we have that

$$
V(x_0 + (1-\lambda)h, y_0 - h) \le V(x_0, y_0),
$$

where h is an arbitrary positive real number. Allowing $h \to 0$ we obtain that

$$
\begin{aligned}
& \lim_{h \to 0} \frac{1}{h} \big[V(x_0 - (1+\lambda)h, y_0 + h) - V(x_0, y_0) \big] = -(1+\lambda) V_x(x_0, y_0) + V_y(x_0, y_0) \le 0 \\
& \lim_{h \to 0} \frac{1}{h} \big[V(x_0 + (1-\lambda)h, y_0 - h) - V(x_0, y_0) \big] = (1-\lambda) V_x(x_0, y_0) - V_y(x_0, y_0) \le 0
\end{aligned}
$$

Again, since the point (x_0, y_0) was chosen arbitrary, this holds for every point (x, y) in the domain of V. □

Even though the two different formulations of the same problem, (2.3) and (2.14), yield the same result, the latter gives more explicit implications

[9]Here we use the fact that the value function at any arbitrarily chosen point along the Buy or the Sell line is less or equal to the maximum attainable value.

for the practical realization of a numerical procedure, which, in addition, becomes more robust. This procedure could also be successfully applied in the case with no transaction costs when a closed-form solution is not attainable.

Recall the definition (2.13) of the NT region in the framework of the stochastic impulse control theory. Note that in the case of no transaction costs and in the case of proportional transaction costs only, we expect that $\mathcal{M}V - V = 0$ everywhere on the domain of the value function, since the value function is continuous in the direction of transaction along the Buy or the Sell line. This is unlike the case with a fixed cost component, where an initial infinitesimal change in y results in a jump in x. As a result, the maximum $(x', y') := \arg\max \mathcal{M}V(x, y)$ lies on the Buy or Sell line[10] with origin at $(x - k, y)$. This maximum might be less than the original value at (x, y). That is why one requires $\mathcal{M}V - V \leq 0$ everywhere. But again, as the fixed costs vanish, that is, $k \to 0$, we obtain equality in the limit. This, in particular, breaks down the definition of the NT region (2.13) used in the stochastic impulse control theory when we try to apply it for the case with no fixed costs. Consequently, it makes sense to redefine the maximum utility operator as

$$\mathcal{M}V(t, x, y) = \sup_{(x',y') \in \mathcal{A}(x,y), (x',y') \neq (x,y)} V(t, x', y'). \tag{3.4}$$

Note that in the new definition of the maximum utility operator we require that $(x', y') \neq (x, y)$. That is, in finding the best possible transaction we do not consider the initial point and require a non-zero (probably infinitesimal) transaction size. Now, with a new definition of the maximum utility operator, the problem with only proportional transaction costs can be correctly characterized by means of QVI (2.14).

3.2 A Unified Framework

Recall that the optimal portfolio choice problem is defined by (1.4) where the net wealth is defined by (1.3). The transaction costs are linear in the size of trade and are defined by (3.1). Some possible linear specifications of

[10] Considering now changes in (x, y) caused by proportional transaction costs only.

the transaction costs structure are

$$f(d\xi) = \begin{cases} 0 & \text{no transaction costs,} \\ \lambda|d\xi| & \text{proportional transaction costs,} \\ I_{\{d\xi\neq 0\}}k & \text{fixed transaction costs,} \\ I_{\{d\xi\neq 0\}}k + \lambda|d\xi| & \text{fixed and proportional transaction costs.} \end{cases}$$

Our approach is also fully applicable in cases where transaction costs are a piecewise-linear function in the size of trade. The simplest example of piecewise-linear transaction costs is given by

$$f(d\xi) = \begin{cases} I_{\{d\xi\neq 0\}}k + \lambda_1|d\xi| & \text{if } |d\xi| \leq \pi, \\ I_{\{d\xi\neq 0\}}k + \lambda_1\pi + \lambda_2(|d\xi| - \pi) & \text{if } |d\xi| > \pi, \end{cases}$$

where π is some threshold size of trade such that the investor pays proportional transaction costs λ_1 when the size of trade is less than π, and pays proportional transaction costs λ_2 on that fraction of trade which exceeds π. With a piecewise-linear function in the size of trade one can model the realistic transaction costs structure which is non-linear. Such a realistic transaction costs structure can include decreasing commissions when transaction volume increases, and/or increasing marginal market impact costs to reflect illiquidity when transaction volume exceeds some particular size.

Now we turn on to presentation of our unified approach. Even though in the presence of transaction costs the control is generally discontinuous, the problem of the choice of an optimal action arises at every time instant. This idea prompts us to formulate the Bellman principle of optimality as

$$V(t-, x(t-), y(t-)) = \max_{d\xi_t} V(t, x(t), y(t)) = E_t\big[V(t+dt-, x(t+dt-), y(t+dt-))\big], \tag{3.5}$$

considering the time interval $[t, t+dt)$ and assuming we know the value function at the next time instant $t+dt-$. This means that the control $d\xi_t$ applies at time $t-$ so that

$$\begin{aligned} y(t) &= y(t-) + d\xi_t, \\ x(t) &= x(t-) - d\xi_t - f(d\xi_t). \end{aligned} \tag{3.6}$$

The evolutions of the amounts invested in the stock and in the bank during the time interval $(t, t+dt)$ are given by

$$\begin{aligned} dy_t &= y_t \mu dt + y_t \sigma dB_t, \\ dx_t &= x_t r dt. \end{aligned} \tag{3.7}$$

Now if we combine (3.7) and (3.6), the dynamics of x_t and y_t during the time interval $(t, t+dt]$ takes the following form

$$\begin{aligned} dy_t &= y_t \mu dt + y_t \sigma dB_t + d\xi_t, \\ dx_t &= x_t r dt - d\xi_t - f(d\xi_t). \end{aligned} \tag{3.8}$$

Note that in (3.8) the control $d\xi_t$ is applied at $t + dt-$ and chosen to solve the dynamic program for the consequent time interval, say, $(t+dt, t+2dt]$. The process

$$\xi(t) = \int_0^t d\xi(s)$$

represents cumulative transaction of the stock up to time t. The process $\xi(t)$ is a right-continuous with left-hand limits (RCLL) $\{\mathcal{F}_t\}$-adapted processes. By convention, $\xi(0) = 0$.

To make it more rigorous, we need to distinguish between continuous and discontinuous parts of $\xi(t)$. We denote the continuous part of $\xi(t)$ by

$$\xi^c(t) \stackrel{\text{def}}{=} \xi(t) - \sum_{0 \le s^d \le t} (\xi(s^d) - \xi(s^d-)),$$

where s^d denotes the times when the control is discontinuous. Furthermore, we need to distinguish between the positive $\xi^{c+}(t)$ and the negative $\xi^{c-}(t)$ parts of $\xi^c(t)$

$$\xi^c(t) = \xi^{c+}(t) - \xi^{c-}(t),$$

such that $d\xi^{c+}(t)$ and $d\xi^{c-}(t)$ are both positive processes and correspond to $dM^c(t)$ and $dL^c(t)$ in the model of Davis and Norman (1990).

We now proceed further to the characterization of the value function. By analogy with the stochastic impulse control theory we make use of the maximum utility operator $\mathcal{M}$:

$$\mathcal{M}V(t, x, y) = \sup_{d\xi_t \neq 0} V(t, x - d\xi_t - f(d\xi_t), y + d\xi_t), \tag{3.9}$$

where $d\xi_t$ belongs to the set of admissible controls available to the investor who starts at time t with an amount of x in the bank and y holdings in the stock. The Bellman principle of optimality (3.5) could be rewritten now as

$$V(t,x,y) = \max\left\{\mathcal{M}V(t,x,y), E_t\big[V(t+dt, x+xrdt, y+y\mu dt+y\sigma dB_t)\big]\right\}. \tag{3.10}$$

This says that at every time t the value function at the state (x,y) equals to the maximum value attainable of choosing either the best immediate transaction or doing nothing (when in the optimum we get $d\xi_t = 0$). In the latter case, if there exists a sufficiently regular solution for the value function, the application of the Ito's rule gives us $\mathcal{L}V(t,x,y) = 0$, where the operator $\mathcal{L}$ is defined by (2.2). Generally, the value function must be not less than the expectation of the value function at $t + dt$, that is

$$V(t,x,y) \geq E_t\big[V(t+dt, x+xrdt, y+y\mu dt+y\sigma dB_t)\big],$$

which implies that

$$\mathcal{L}V(t,x,y) \leq 0 \quad \text{in } [0,T]\times\mathcal{S}. \tag{3.11}$$

The definition of the no transaction region remains the same as in the framework of the impulse control theory:

$$D = \big\{(x,y) : V(t,x,y) > \mathcal{M}V(t,x,y)\big\}. \tag{3.12}$$

Note, however, that our definition of the maximum utility operator is slightly different. Finally, we can rewrite (3.10) as

$$\max\{\mathcal{M}V(t,x,y) - V(t,x,y), \mathcal{L}V(t,x,y)\} = 0 \quad \text{in } [0,T]\times\mathcal{S}, \tag{3.13}$$

which says that at least one of the terms in (3.13) must hold with an equality.

The following theorem characterizes the value function.

Theorem 3.2 (Verification Theorem 3). *Suppose there exists a function $v(t,x,y) \in C^{1,1,2}$ that satisfies the growth conditions and an admissible control $\xi(t)$ such that*

$$\max\Big\{\mathcal{L}v, \quad \mathcal{M}v - v\Big\} = 0 \quad \textit{on } [0,T]\times\mathcal{S},$$

$$\mathcal{M}v - v = 0 \quad \textit{outside } D, \tag{3.14}$$

$$\mathcal{L}v(t,x,y) = 0 \quad in\ D, \tag{3.15}$$

$$v(T,x,y) = U(X_T), \tag{3.16}$$

then

$$v(t,x,y) = V(t,x,y),$$

and the control $\xi(t)$*, which is given by*

$$d\xi_t := \arg\max\big\{v(t, x - d\xi_t - f(d\xi_t), y + d\xi_t)\big\},$$

is optimal.

Proof. Using the Ito's rule for semimartingales we obtain

$$\begin{aligned}
E_t^{x,y}[v(T,x_T,y_T)] = v(t,x_t,y_t) &+ \int_t^T \mathcal{L}v(s,x,y)ds \\
&+ \int_t^T [-(1+\lambda)v_x + v_y]d\xi_s^{c+} + \int_t^T [(1-\lambda)v_x - v_y]d\xi_s^{c-} \\
&+ \sum_{t\le s^d\le T} [v(s^d, x_{s^d}, y_{s^d}) - v(s^d-, x_{s^d-}, y_{s^d-})].
\end{aligned} \tag{3.17}$$

The second and the third terms in (3.17) are due to the continuous control. That is, when the control is continuous[11]

$$\begin{aligned}
&v(s,x_s,y_s) - v(s-,x_{s-},y_{s-}) \\
&\quad = v(s, x_{s-} - d\xi_s^c - f(d\xi_s^c), y_{s-} + d\xi_s^c) - v(s-,x_{s-},y_{s-}) \\
&\quad = \begin{cases} [-(1+\lambda)v_x + v_y]d\xi^{c+}(s) & \text{when } d\xi_s^c > 0, \\ -[(1-\lambda)v_x - v_y]d\xi^{c-}(s) & \text{when } d\xi_s^c < 0. \end{cases}
\end{aligned}$$

Note that the transactions are maid in order to maximize the expected utility, that is $v(s,x_s,y_s) = \mathcal{M}v(s-,x_{s-},y_{s-})$ for both the continuous and discontinuous controls. Then, using (3.14) we obtain

$$v(s,x_s,y_s) - v(s-,x_{s-},y_{s-}) = \mathcal{M}v(s-,x_{s-},y_{s-}) - v(s-,x_{s-},y_{s-}) = 0.$$

Consequently, when transactions take place continuously in time, the fol-

[11]Note that the control is continuous only if $k = 0$.

lowing conditions must hold

$$\begin{aligned} [-(1+\lambda)v_x + v_y]d\xi^{c+}(t) &= 0 \quad \text{for all } t, \\ [(1-\lambda)v_x - v_y]d\xi^{c-}(t) &= 0 \quad \text{for all } t. \end{aligned} \tag{3.18}$$

In addition, when the transactions are discontinuous

$$\sum_{t\le s^d\le T} [v(s^d, x_{s^d}, y_{s^d}) - v(s^d-, x_{s^d-}, y_{s^d-})] = 0. \tag{3.19}$$

Finally, using (3.18), (3.19), (3.15) and (3.16) we get

$$v(t, x_t, y_t) = E_t^{x,y}[v(T, x_T, y_T)] = V(t, x_t, y_t). \qquad \square$$

Remark 3.1. If $k = 0$ and the transaction costs structure is linear, the control is continuous and equation (3.17) recovers the model of Davis and Norman (1990).

Remark 3.2. When the transaction costs structure is linear but the stock follows a jump-diffusion process, the control is a combination of continuous and discontinuous controls (see Framstad et al. (2001)), even if $k = 0$.

Remark 3.3. When the transaction costs structure is piecewise-linear, the optimal control is generally a combination of continuous and discontinuous controls[12], even if $k = 0$. Consequently, the problem cannot be correctly formulated within the framework of Davis and Norman (1990).

Remark 3.4. If $k > 0$, then all transactions involve jumps. That is, if the transaction costs include a fixed component, then the infinitesimal transaction policy is not optimal. In this case $\xi^c(t) \equiv 0$, and the problem largely amounts to a classical impulse control problem. However, as $k \to 0$, the model with both fixed and proportional transaction costs converges correctly to the model with proportional transaction costs only.

The above given verification theorem can be used constructively in the search for the value function. The analytical and numerical constructions of the value function are fulfilled in the same manner as in the framework of the stochastic impulse control theory.

As a converse to the verification theorem one can prove the viscosity property of the value function.

[12]To get some insight into the nature of optimal transaction policy see Demchuk (2002).

Theorem 3.3. *The value function V defined by (1.4), assuming it is continuous in $[0, T] \times \mathcal{S}$, is a viscosity solution of (3.13).*

Proof. The proof is similar to that of Theorem 2.4. $\square$

Theorem 3.3 states that the solution to the problem (1.4) exists. In addition, the uniqueness of the solution can be proved in the same manner as in Theorem 3.8 (Comparison Theorem with subsequent Corollary) of Øksendal and Sulem (2002). The technical meaning of the existence and uniqueness of the solution is that this solution can be computed by standard discretization methods. We will review these methods in the subsequent section.

4 Numerical Methods for Optimal Portfolio Choice Problem with Transaction Costs

In this section we outline some methods for solving numerically the optimal portfolio choice problem with a linear structure of transaction costs. The general solution method is based on the proposed unified theoretical framework presented in the preceding section. The objective is to find the value function $V(t, x, y)$, which is characterized by QVI (3.13).

There are two basic approaches to the solution of continuous-time continuous-space stochastic control problems. The first one is based on the idea of approximating PDEs by finite differences, and the second approach involves a consistent approximation of the problem by a Markov chain, and then the solution of an appropriate optimization problem for the Markov chain model. In the following two subsections we review both these methods.

An upright implementation of the general solution method is extremely time consuming. In the third subsection we show how the computational time can be substantially reduced by exploiting the knowledge of the form of the optimal portfolio strategy. In addition, we also present some alternative numerical methods that use certain gradient constraints.

4.1 A General Finite Difference Method

Using the concept of viscosity solutions, in particular the stability property, the general theory of Barles and Souganides (1991) provides a framework for proving the uniform convergence of numerical schemes. We refer to Barles (1997) for a thorough discussion of the convergence of numerical schemes with applications to finance. This framework yields, after checking its assumptions, the convergence of a finite difference scheme for a PDE. In particular, one requires that a scheme satisfy the following conditions: stability, consistency, and monotonicity.

To solve the problem using a finite difference approximation, we first localize the problem on the bounded space $(t,T)\times(X_{min},X_{max})\times(Y_{min},Y_{max})$. Then we define the finite difference grid $(n\delta t, i\delta x, j\delta y)$ on this space. Afterwards, the localized problem on the finite difference grid is solved by using the following backward recursion algorithm: First, we solve the partial differential equation $\mathcal{L}V(t,x,y)=0$ between times $n\delta t$ and $(n+1)\delta t$. By doing this we find the value function at each point $(n\delta t, x, y)$ assuming no transactions. That is, we find

$$V(n\delta t, x(n\delta t), y(n\delta t)) = E[V(n\delta t+\delta t, x(n\delta t+\delta t), y(n\delta t+\delta t))], \tag{4.1}$$

which is a lower estimate for the value function. Then we need to compare the value function at each point $(n\delta t, x, y)$ with the maximum attainable values from either buying or selling some amount of the stock. That is, we perform

$$\begin{aligned} V(n\delta t, x, y) = \max\Big\{ & \max_m V(n\delta t, x - m\delta y - f(m\delta y), y + m\delta y), \\ & \max_m V(n\delta t, x + m\delta y - f(m\delta y), y - m\delta y)\Big\}, \end{aligned} \tag{4.2}$$

where m runs through the positive integer numbers $(m = 0, 1, 2, \ldots)$. Note that we can combine (4.1) and (4.2) into one equation defining the solution algorithm

$$\begin{aligned} V(n\delta t, x, y) = \max\Big\{ & \max_m V(n\delta t, x - m\delta y - f(m\delta y), y + m\delta y), \\ & \max_m V(n\delta t, x + m\delta y - f(m\delta y), y - m\delta y), \\ & E[V(n\delta t+\delta t, x(n\delta t+\delta t), y(n\delta t+\delta t))]\Big\}, \end{aligned} \tag{4.3}$$

where $m = 1, 2, \ldots$. The explicit finite difference scheme for solving the partial differential equation $\mathcal{L}V(t,x,y)=0$ is based on the following commonly used finite difference approximations for the derivatives:

$$V_t(n\delta t, i\delta x, j\delta y) \approx \frac{V(n\delta t, i\delta x, j\delta y) - V((n-1)\delta t, i\delta x, j\delta y)}{\delta t}$$

$$V_x(n\delta t, i\delta x, j\delta y) \approx \begin{cases} \frac{V(n\delta t,(i+1)\delta x, j\delta y) - V(n\delta t, i\delta x, j\delta y)}{\delta x} & \text{when } x \geq 0 \\ \frac{V(n\delta t, i\delta x, j\delta y) - V(n\delta t,(i-1)\delta x, j\delta y)}{\delta x} & \text{when } x < 0 \end{cases}$$

$$V_y(n\delta t, i\delta x, j\delta y) \approx \frac{V(n\delta t, i\delta x, (j+1)\delta y) - V(n\delta t, i\delta x, (j-1)\delta y)}{2\delta y}$$

$$V_{yy}(n\delta t, i\delta x, j\delta y) \approx \frac{V(n\delta t, i\delta x, (j+1)\delta y) - 2V(n\delta t, i\delta x, j\delta y) + V(n\delta t, i\delta x, (j-1)\delta y)}{\delta y^2}$$

Because there is no diffusion in the x direction, the choice of approximation is important. For this reason, a one-sided difference must be used. If x changes sign, then the choice of difference must reflect this.

In every type of approximation one faces the problem of specifying boundary conditions. The values of, for example, $V(n\delta t, i\delta x, Y_{min})$ and $V(n\delta t, i\delta x, Y_{max})$ can be found by either extrapolation from interior points or one-sided differencing. An alternative choice is to shrink in time the region spanned by computational grid.

4.2 A Markov Chain Approximation Method

The other method of solution of such problems was suggested by Kushner (see, for example, Kushner and Martins (1991)). First, according to the Markov chain approximation method, one constructs discrete time approximations of the continuous time price processes used in the continuous time model. Then the discrete time program is solved by using the discrete time dynamic programming algorithm (i.e., backward recursion algorithm). In a practical application of this approach one often discretizes a PDE by applying the finite-difference approximation scheme which serves here as a guide to the construction of a Markov chain. The coefficients of the resulting discrete equation is then used as the transition probabilities. Thus, the finite difference scheme is given a probabilistic interpretation.

The simplest Markov chain approximation of the optimal portfolio choice problem could be as described below. Consider the partition $0 = t_0 < t_1 < \ldots < t_n = T$ of the time interval $[0, T]$ and assume that $t_i = i\Delta t$ for

$i = 0, 1, \ldots, n$ where $\Delta t = \frac{T}{n}$. Let ε be a stochastic variable:

$$\varepsilon = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } 1-p. \end{cases}$$

We define the discrete time stochastic process of the stock as

$$S_{t_{i+1}} = S_{t_i}\varepsilon, \tag{4.4}$$

and the discrete time process of the risk-free asset as

$$x_{t_{i+1}} = x_{t_i}\rho. \tag{4.5}$$

If we choose $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}\left[1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right]$, we obtain the binomial model proposed by Cox, Ross, and Rubinstein (1979). An alternative choice is $u = e^{(\mu-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}}$, $d = e^{(\mu-\frac{1}{2}\sigma^2)\Delta t-\sigma\sqrt{\Delta t}}$, $\rho = e^{r\Delta t}$, and $p = \frac{1}{2}$, which was proposed by He (1990). As n goes to infinity, the discrete time processes (4.4) and (4.5) converge in distribution to their continuous counterparts (1.2) and (1.1). This is what is called the *local consistency conditions* for a Markov chain.

The following discretization scheme is proposed to find the value function $V(t, x, y)$ defined by (3.13)

$$\begin{aligned} V^{\Delta t}(t_i, x, y) = \max\Big\{ &\max_m V^{\Delta t}(t_i, x - m\delta y - f(m\delta y), y + m\delta y), \\ &\max_m V^{\Delta t}(t_i, x + m\delta y - f(m\delta y), y - m\delta y), \\ &E[V^{\Delta t}(t_{i+1}, x\rho, y\varepsilon)]\Big\}, \end{aligned} \tag{4.6}$$

where m runs through the positive integer numbers ($m = 1, 2, \ldots$), and

$$\begin{aligned} &V^{\Delta t}(t_i, x - m\delta y - f(m\delta y), y + m\delta y) \\ &\qquad = E\left\{V^{\Delta t}(t_{i+1}, (x - m\delta y - f(m\delta y))\rho, (y + m\delta y)\varepsilon)\right\} \\ &V^{\Delta t}(t_i, x + m\delta y - f(m\delta y), y - m\delta y) \\ &\qquad = E\left\{V^{\Delta t}(t_{i+1}, (x + m\delta y - f(m\delta y))\rho, (y - m\delta y)\varepsilon)\right\}, \end{aligned}$$

if at time t_i we do not know yet the value function. Here we have discretized the y-space in a lattice with grid size δy, and the x-space in a lattice with grid size δx[13]. This scheme is a dynamic programming formulation of the discrete time problem. The solution procedure is as follows: Start at the terminal date and give the value function values by using the boundary conditions as for the continuous value function over the discrete state space. Then work backwards in time. That is, at every time instant t_i and every particular state (x, y), by knowing the value function for all the states in the next time instant, t_{i+1}, find the investor's optimal policy. This is carried out by comparing maximum attainable utilities from buying, selling, or doing nothing. Note that the proposed algorithm employing a Markov discretization scheme (4.6) is completely equivalent to a proposed algorithm using a finite difference scheme (4.3). The only difference is in how we find the expectation of the value function. Instead of solving the partial differential equation $\mathcal{L}V(t, x, y) = 0$ we find the expectation using

$$E\{V(t_{i+1}, x\rho, y\varepsilon)\} = pV(t_{i+1}, x\rho, yu) + (1-p)V(t_{i+1}, x\rho, yd).$$

From a computational point of view the Markov chain approximation approach is sometimes easier to implement, because one does not face the issue of the stability of a finite difference scheme. Consequently, even for a relatively high Δt one gets a reasonable estimate for the value function and the associated optimal policy.

Theorem 4.1. *The solution $V^{\Delta t}$ of (4.6) converges weakly to the unique viscosity solution of the continuous time problem characterized by (3.13) as $\Delta t \to 0$.*

For a rigorous treatment of a proof of this type of convergence theorems, we refer the reader to, for example, Kushner and Martins (1991), Davis et al. (1993), and Davis and Panas (1994).

Note that in the proposed Markov chain approximation method, unlike the finite-difference approximation, in the calculation of the value function on the $(i\delta x, j\delta y)$ grid at timestep $n\delta t$ one generally needs to know the value function outside of the grid points at timestep $(n+1)\delta t$. The required values can be estimated by two dimensional interpolation. Note, however, that even in the finite-difference approximation method we cannot avoid the two dimensional interpolation: In performing the search for a maximum along

[13] It is supposed that $\lim_{\Delta t \to 0} \delta y \to 0$, and $\lim_{\Delta t \to 0} \delta x \to 0$, that is, $\delta y = c_y \Delta t$, and $\delta x = c_x \Delta t$ for some constants c_y and c_x.

the direction of transaction starting at some node $(i\delta x, j\delta y)$ we generally go past the grid points.

4.3 Practical Solution Methods

So far the outputs of a general solution method are the value function and the optimal transaction policy described as the mapping $(x, y) \mapsto (x', y')$. That is, we implicitly assumed that for every point $(x, y) \in (i\delta x, j\delta y)$ the algorithm finds a new point $(x', y') \in (i\delta x, j\delta y)$ that represents the optimal transaction. An upright implementation of such an algorithm is extremely time consuming. In this subsection we show how the computational time can be substantially reduced by exploiting the knowledge of the form of the optimal portfolio strategy. Besides, here we present some alternative numerical methods which use certain gradient constraints and could be implemented not only using a finite difference type of approximation, but also employing a Markov chain approximation. For the ease of the exposition we assume that in the presence of transaction costs the Buy, Sell, and NT regions have no subregions. However, one should be aware of the fact that it is not always the case and every region might consist of some subregions. In this situation the solution methods are more complicated as they must take into account the possible presence of several subregions.

First, we consider the case with proportional transaction costs only. We know that at every time t the optimal policy could be described by two equations $y = y_l(x)$ and $y = y_u(x)$ which define the lower and the upper boundaries, respectively, of the NT region. The proposed trading strategy is to transact immediately to the nearest boundary if the portfolio lies outside of the NT region. This means that

$$\mathcal{M}V(t, x, y) = \begin{cases} V(t, x + (1-\lambda)(y - y_u), y_u) & \text{if } y \geq y_u(x), \\ V(t, x - (1+\lambda)(y_l - y), y_l) & \text{if } y \leq y_l(x). \end{cases} \tag{4.7}$$

The first order conditions of optimality of y_l and y_u give

$$\begin{aligned} (1-\lambda)V_x(t, x', y_u) - V_y(t, x', y_u) = 0, \\ -(1+\lambda)V_x(t, x', y_l) + V_y(t, x', y_l) = 0. \end{aligned} \tag{4.8}$$

As an additional source of information we know that inside the NT region we must have

$$
\begin{aligned}
(1-\lambda)V_x(t,x,y) - V_y(t,x,y) &\le 0, \\
-(1+\lambda)V_x(t,x,y) + V_y(t,x,y) &\le 0,
\end{aligned}
\tag{4.9}
$$

as rebalancing the portfolio is not optimal here. The solution procedure could be implemented as follows: First, start from the final date and solve $\mathcal{L}V(t,x,y) = 0$ to find the lower estimate of the value function at the preceding time instant. Then for every x find y_l and y_u such that in the points (x, y_l) and (x, y_u) the conditions (4.8) are satisfied. The optimal amount y^* without transaction costs is the natural start for the search of the points that satisfy the gradient constraints. Afterwards, recompute the value function outside the no transaction region using $V(t,x,y) = \mathcal{M}V(t,x,y)$ (see equations (4.7)). It could be easily checked that after the recomputation we will have

$$
\begin{aligned}
(1-\lambda)V_x(t,x,y) - V_y(t,x,y) &= 0 \quad \text{in the Sell region}, \\
-(1+\lambda)V_x(t,x,y) + V_y(t,x,y) &= 0 \quad \text{in the Buy region}.
\end{aligned}
\tag{4.10}
$$

Repeat the previous steps for the remaining time instants backwards to the initial date.

The realization of the solution method described above for a Markov chain approximation approach is somewhat different. Davis et al. (1993), and Davis and Panas (1994) propose the following algorithm for this method

$$
\begin{aligned}
V^{\Delta t}(t_i,x,y) = \max\Big\{ & V^{\Delta t}(t_i, x-(1+\lambda)\delta y, y+\delta y), \\
& V^{\Delta t}(t_i, x+(1-\lambda)\delta y, y-\delta y), \\
& E\{V^{\Delta t}(t_{i+1}, x\rho, y\varepsilon)\}\Big\},
\end{aligned}
\tag{4.11}
$$

where they replace the gradient constraints (4.9) by

$$
\begin{aligned}
V^{\Delta t}(t_i, x-(1+\lambda)\delta y, y+\delta y) &\le V^{\Delta t}(t_i,x,y), \\
V^{\Delta t}(t_i, x+(1-\lambda)\delta y, y-\delta y) &\le V^{\Delta t}(t_i,x,y).
\end{aligned}
$$

It is straightforward to show that the solution of (4.11) converges to the solution of variational inequalities (2.3) as $\Delta t \to 0$. However, an upright application of the numerical scheme (4.11) gives the correct result only if one happens to start from a point inside the NT region. That is, this scheme gives only implicit indications for the construction of a usable algorithm. The practical implementation of such an algorithm (see Davis and Panas (1994) Section 5) starts with the search of the boundaries of the NT region. A schematic computer program of their algorithm for the search of the lower boundary of the NT region for a fixed t and x can be written as follows

```
for j=0:jMax
   y=j*δy;
   if V(t,x-(1+λ)(y+δy),y+δy)>V(t,x-(1+λ)y,y)
      continue;
   else
      break;
   end;
end;
```

Simply put, starting from the point $(t, x, 0)$ outside of the NT region (assuming $\mu > r$) the algorithm searches for the first point (t, x', y_l) which satisfies the second inequality[14] in (4.9). A similar algorithm is used to search for the upper boundary of the NT region. That is, starting from the point (t, x, y_{max}) outside of the NT region, the algorithm searches for the first point (t, x', y_u) which satisfies the first inequality in (4.9). The value function outside the NT region is determined in accordance with (4.7).

In addition, in the case with only proportional transaction costs and the negative exponential utility function, the value function is C^2 everywhere and has only one maximum along the direction of transaction. This suggests itself to improve the algorithm of Davis and Panas (1994) by using some sort of Newton-Raphson method to find the roots of the ordinary differential equations that represent the gradient constraints.

Now we proceed further to the case where transaction costs have both fixed and proportional components. We conjecture that in this case the optimal policy could be described by four boundaries: $y = y_u(x)$ and $y = y_l(x)$, which describe the upper and the lower boundaries of the NT region, and $y = y_u^*(x)$ and $y = y_l^*(x)$ which describe the Sell and the Buy target boundaries. The proposed trading strategy is to transact immediately to

[14] It is easy to see that an alternative interpretation of this algorithm could be the search of maximum as one transacts along the Buy direction.

the Buy target boundary if the portfolio lies in the Buy region, i.e., when $y < y_l(x)$, or to the Sell target boundary if the portfolio lies in the Sell region, $y > y_u(x)$. This could be expressed as follows

$$\mathcal{M}V(t,x,y) = \begin{cases} V(t, x-k+(1-\lambda)(y-y_u^*), y_u^*) & \text{if } y \geq y_u(x), \\ V(t, x-k-(1+\lambda)(y_l^*-y), y_l^*) & \text{if } y \leq y_l(x). \end{cases} \tag{4.12}$$

The first order conditions of optimality of y_l^* and y_u^* give

$$\begin{aligned} (1-\lambda)V_x(t,x',y_u^*) - V_y(t,x',y_u^*) = 0, \\ -(1+\lambda)V_x(t,x',y_l^*) + V_y(t,x',y_l^*) = 0. \end{aligned} \tag{4.13}$$

Besides, the conditions (4.12) are valid including the boundaries of the no transaction region. Thus, we have the following two equations

$$\begin{aligned} V(t,x,y_u) = V(t, x-k+(1-\lambda)(y_u-y_u^*), y_u^*), \\ V(t,x,y_l) = V(t, x-k-(1+\lambda)(y_l^*-y_l), y_l^*). \end{aligned} \tag{4.14}$$

As in the case with proportional transaction costs only, it is easy to show that (4.10) will hold.

In principle, we could try to implement the solution procedure as follows: First solve $\mathcal{L}V(t,x,y) = 0$. Then for every x find y_l^* and y_u^* such that in the points (x, y_l^*) and (x, y_u^*) the conditions (4.13) are satisfied. Proceed further to find y_l and y_u using the link given by equations (4.14). Finally, recompute the value function using $V(t,x,y) = \mathcal{M}V(t,x,y)$ given by (4.12). The practical implementation of this procedure is generally not feasible due to a couple of reasons: First, the problem is that the conditions (4.13) give us local maxima, not global ones. Unlike the case with proportional transaction costs only, we are not sure that everywhere inside the no transaction region the conditions (4.9) hold. Second, multiple local maxima along the direction of transaction can be caused by a coarse grid in conjunction with the presence of a fixed cost component. At least, we observed such a behavior of the value function for the power utility function. Thus, in the case where each transaction have a fixed cost component, the approach based on the maximum utility operator is preferable.

Note that the case with both fixed and proportional transaction costs is much more complicated than that with only proportional transaction costs:

The optimal strategy is described by four instead of two boundaries. The search algorithms that do not use derivatives are preferable as they are more robust. Moreover, one may face the problem with multiple local maxima along the direction of transaction, and there are no good ways to find the global one. A standard heuristic that is used: find local maxima and then continue the search further along the direction of transaction. All these result in a drastic reduction of computational speed as compared to a similar problem with proportional transaction costs only. A practical algorithm for this case is based on the idea to implement the maximum utility operator $\mathcal{M}V(t,x,y)$ as a function `MaxUtilityOp(t,x,y,newy)`, which returns `true` when the optimal strategy for (t,x,y) is to transact, and `false` otherwise. In the former case (if the function returns `true`) the variable `newy` contains the target amount in y. To find the maximum along the direction of transaction, one needs first to implement a routine for initially bracketing a maximum, and then to implement either the classical *bracketing* algorithm or the *golden section search* algorithm to find the maximum.

Having implemented the function `MaxUtilityOp`, one can proceed to the search of the boundaries of the NT region. A schematic computer program of the *bisection* algorithm for the search of the lower boundary of the NT region and the Buy target boundary for a fixed t and x can be implemented as follows

```
a=0;
b=yMax;
while (b-a) > δy
   c=(b+a)/2;
   if MaxUtilityOp(t,x,c,newy)
      a=c;
   else
      b=c;
end;
```

It is supposed that the point $(t,x,0)$ lies in the Buy region and the point (t,x,y_{max}) lies inside the NT region. Bisection proceeds by evaluating the maximum utility operator at the midpoint of the original interval $c = (b+a)/2$ and testing to see in which of the subintervals $[a,c]$ or $[c,b]$ the boundary of the NT region lies. The procedure is then repeated with the new interval as often as needed to locate the solution with the desired accuracy. A similar algorithm could be used to search for the upper boundary of the NT region and the Sell target boundary. The value function outside the NT region is determined in accordance with (4.12).

However, the above presented algorithm works acceptably fast for only the negative exponential utility, where it is possible to reduce the dimensionality of the problem. For a general utility function the algorithm is prohibitive slow for practical applications. It is a challenging and really concerning issue to find the ways to increase the computational speed of the numerical algorithm. Moreover, it is important to develop efficient numerical methods to handle the problem with many stocks.

Since there are almost no closed-form solutions for the optimal portfolio selection problem with transaction costs and the numerical methods are computationally hard, for practical applications it is of major importance to use other alternatives such as asymptotic solutions and approximation. The asymptotic analysis was first presented by Atkinson and Wilmott (1995) and then successfully applied in different contexts by Whalley and Wilmott (1997), Barles and Soner (1998), Korn (1998), and Janeček and Shreve (2004). The asymptotic analysis method studies the limiting behavior of the value function and the associated optimal trading policy as one or several parameters go to zero. However, as far as we know, no one has ever compared the difference between the exact numerical and asymptotic solutions under realistic parameters. In addition, finding a reasonable approximate[15] solution is an interesting subject for further research. Approximation methods have never been studied in the context of the optimal portfolio selection problem with transaction costs.

5 Conclusions and Extensions

In this paper we studied the continuous time optimal portfolio selection problem for an investor with a finite horizon who maximizes expected utility of terminal wealth and faces transaction costs in the capital market. At present, depending on a particular structure of transaction costs, such a problem is formulated within the framework of either singular stochastic

[15] Under approximation we mean finding a function that is "close to" the exact numerical solution over some interval. As a measure of goodness of fit one usually uses a L^p norm.

control or impulse stochastic control theory. In this paper we suggested a unified theoretical framework, which generalizes the contemporary approaches and is capable to deal with any problem where transaction costs are a linear/piecewise-linear function of the amount of the risky asset traded. Mainly, our idea was to integrate the stochastic singular and impulse control theories into a single approach. We also discussed some methods for solving numerically the optimal portfolio selection problem within our unified framework.

The approach of this paper may be easily extended to the infinite horizon. Our approach may be also generalized in a straightforward manner to incorporate intermediate consumption, jump diffusion processes, and several risky assets. Generalization to include charging transaction costs in both the risky and riskless assets is also easy to accommodate. The approach is also applicable to similar problems in discrete time. Moreover, in a discrete time model the presented approach is not limited to the cases where transaction cost are linear/piecewise-linear, but could be also successfully applied to problems with fully non-linear transaction costs.

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